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Equations and a fast algorithm for determining the probability of failure initiated by flaws

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Abstract

Powerful equations and an efficient algorithm are proposed for determining the probability of failure of loaded components with complex shape, containing multiple types of flaws. The equations are based on the concept 'conditional individual probability of initiating failure' characterising a single flaw *given* that it is in the stressed component. The proposed models relate in a simple fashion the conditional individual probability of failure characterising a single flaw (estimated by a Monte Carlo simulation) to the probability of failure characterising a population of flaws. The derived equations constitutes the core of a new statistical theory of failure initiated by flaws in the material, with important applications in optimising designs by decreasing their vulnerability to failure initiated by flaws during overloading or fatigue cycling.

Methods have also been developed for specifying the maximum acceptable level of the flaw number density and the maximum size of the stressed volume which guarantee that the probability of failure initiated by flaws remains below a maximum acceptable level. An important parameter referred to as 'detrimental factor' is also introduced. Components with identical geometry and material, with the same detrimental factors are characterised by the same probability of failure. It is argued that eliminating flaws from the material should concentrate on types of flaws characterised by large detrimental factors.

The equations proposed avoid conservative predictions resulting from equating the probability of failure initiated by a flaw in a stressed region with the probability of existence of the flaw in that region.

Keywords: Flaws; Fracture; Fatigue; Risk; Failure; Statistical theory of failure

1. Background

Early-life failures are often the result of poor manufacturing and inadequate design. A substantial proportion of early-life failures is also due to the presence of flaws in the material.

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Suppose that a component with volume V is subjected to uniaxial tension. In the component, there exist N_σ critical flaws which initiate failure at the loading stress σ. Suppose also that the volume V has been divided into M small zones with volumes ΔV. The probability that no small volume ΔV will contain a critical flaw is

$$(1 - \Delta V/V)^{N_\sigma} \approx 1 - N_\sigma \Delta V/V = 1 - n_\sigma \Delta V$$

where $n_\sigma = N_\sigma/V$ is the flaw number density. The probability p_0 that the entire volume V will survive the loading stress σ with no failure, equals the probability that all small zones with volumes ΔV will survive the stress σ:

$$p_0 = (1 - n_\sigma \Delta V)^M = \exp(M \ln[1 - n_\sigma \Delta V]) \approx \exp(-n_\sigma V) \tag{1}$$

because for $\Delta V \approx 0$, $\ln[1 - n_\sigma \Delta V] \approx -n_\sigma \Delta V$ and $V = M \times \Delta V$.

In order to use Eq. (1), an expression for n_σ is required. Weibull (1951) proposed the empirical relationship

$$n_\sigma V_0 = \left(\frac{\sigma}{\sigma_0}\right)^m \tag{2}$$

where V_0, σ_0 and m are constants. Experimental data related to failure of brittle material conformed well with this assumption (Hull and Clyne, 1996). Given Eq. (2), the probability of failure p_σ of the stressed volume V is determined from the Weibull distribution

$$p_\sigma = 1 - \exp\left(-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right) \tag{3}$$

which is common for describing the strength distribution of materials (Jayatilaka and Trustrum, 1977; Bergman, 1985).

An important factor affecting the strength of components is the presence of flaws due to processing, manufacturing or mechanical damage during service. Currently, most of the existing models relate the probability of failure initiated by defects to the probability of finding a defect of particular size in the stressed volume. Thus, Curry and Knott (1979) and Wallin et al. (1984), in their statistical models for carbide induced brittle fracture in steels, related the probability of brittle fracture to the probability that ahead of the crack tip a carbide will exist, which has a radius greater than some critical value, specified by the Griffith's crack advancement criterion.

Relating the probability of existence of a flaw with critical size to the probability of fracture however, can be made only if the flaws are very weak and initiate fracture easily. In the general case, this approach is overly conservative, because only a small number of flaws of any particular size are liable to initiate failure, even though subjected to high matrix strains. Hahn (1984) pointed out that the crack nucleation on hard particles is assisted by plastic deformation of the surrounding matrix but requires an additional stress raiser or a defect in the particles. Furthermore, to be 'eligible', the particle should have an orientation favourable for nucleating a crack and the misorientations at the particle boundary should produce a low value of the local fracture toughness. All of these requirements are satisfied with certain probability.

It is necessary to point out that that the probability of initiating fracture is also a function of the orientation of the flaws regarding the stress tensor.

Batdorf and Crose (1974) proposed a statistical model for fracture of brittle materials containing randomly oriented microcracks. They demonstrated that for uniaxial tension their theory was equivalent to Weibull's.

Weakest-link theories pertinent to fracture of brittle materials (Evans, 1978) yield a probability of failure Φ given by

$$\Phi(S, V) = 1 - \exp\left[-\int_V \mathrm{d}V \int_0^{K_c} g(v)\,\mathrm{d}v\right] \tag{4}$$

where K_c is the fracture strength, V is a sample volume and $g(v)\,\mathrm{d}v$ is the number of flaws per unit volume with strength between v and $v + \mathrm{d}v$. In fact, the product $\mathrm{d}V \int_0^{K_c} g(v)\,\mathrm{d}v$ gives the number of defects with strength smaller than or equal to K_c, in the infinitesimal volume $\mathrm{d}V$. For the probability of failure p_σ in a volume V with stress σ, Danzer and Lube (1996) proposed the equation

$$p_\sigma = 1 - \exp(-\overline{N}_c) \tag{5}$$

where $\overline{N}_c$ is the expected number of defects with critical size in the stressed volume.

Both Eqs. (4) and (5) are based on the expected number of critical defects (the defects which initiate fracture) in the stressed volume. The number of critical defects in the volume however is not a measurable quantity and is usually unknown.

Using the concept individual probability $F(\sigma)$ of triggering fracture by a single flaw, in earlier work (Todinov, 2000) the probability of failure of a component loaded at a constant stress level σ was determined to be:

$$p_\sigma = 1 - \exp[-\lambda V F(\sigma)] \tag{6}$$

Eq. (6) is based on the assumption that in the stressed volume V, the locations of the random flaws follow a homogeneous Poisson process with constant density $\lambda = \text{const}$. The type of flaws has a strong influence on the probability of failure. Due to tensile tessellation stresses for example, alumina or silicon-based inclusions in steel wire are more likely to become initiators of fracture compared to sulphide inclusions of the same diameter and numbers. In another example, sharp crack-like defects are characterised by a larger probability of initiating fracture compared to defects with globular shape. Furthermore, crack-like defects with a crack plane perpendicular to the direction of the acting tensile stress are more likely to initiate fracture than cracks oriented along the direction of the tensile stress.

If all flaws were critical (initiate failure at a stress level σ) then $F(\sigma) = 1$ and the probability that at least a single flaw will reside in the stressed volume V is

$$p_\sigma = 1 - e^{-\lambda V} \tag{7}$$

which also gives the probability of failure. Since λV is the expected number of critical flaws in the volume V, Eq. (7) is equivalent to Eq. (5).

Next, we will show that Eq. (6) is valid not only for a simple uniaxial stress state. It can also be generalised for a component with complex shape and loading containing flaws.

2. General equation related to the probability of failure of a stressed component with complex shape

Suppose that a component with complex shape is loaded in an arbitrary fashion, and contains non-interacting flaws. It is assumed that the flaws locations in the volume V follow a non-homogeneous Poisson process. The variation of the flaw number density in the volume of the component is described by the function $\lambda(x, y, z)$. It gives the flaw number density in the infinitesimal volume dv at a location with coordinates x, y, z (Fig. 1).

Suppose that a single flaw is characterised by the conditional individual probability F_c of initiating failure *given* that the flaw is present in the stressed component. The index 'c' in F_c means that the individual probability of initiating failure has been conditioned on the existence of a flaw in the component. This probability is

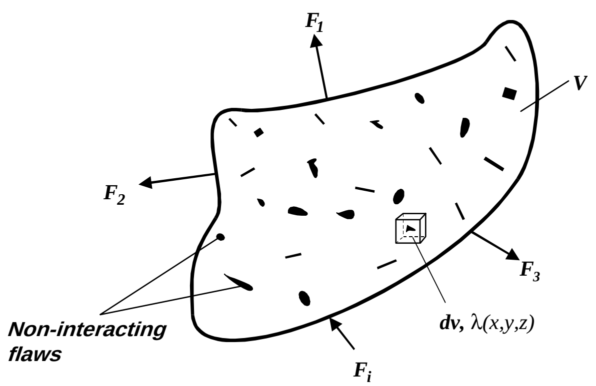

Fig. 1. A component with complex shape, loaded with arbitrary forces F_i.

different from the probability p_f of failure of the component associated with a population of flaws. The probability p_f is related to the whole population of flaws and is not conditioned on the existence of flaws in the component. In other words, p_f is still a valid concept even if flaws are not present at all in the component.

The probability p_f (unconditional) of failure associated with a population of flaws can be determined by subtracting from unity the probability p^0 of the complementary event: 'none of the flaws will initiate failure'. The probability $p^0_{(r)}$ of the compound event: *exactly r flaws exist in the volume V of the component and none of them will initiate failure* is a product

$$p^0_{(r)}(V) = \exp\left(-\int_V \lambda(x,y,z)\,\mathrm{d}v\right)\frac{\left(\int_V \lambda(x,y,z)\,\mathrm{d}v\right)^r}{r!}[1-F_c]^r \tag{8}$$

of the probabilities of two statistically independent events: (i) 'exactly r flaws reside in the volume V', the probability of which is given by the non-homogeneous Poisson distribution

$$P(X=r) = \exp\left(-\int_V \lambda(x,y,z)\,\mathrm{d}v\right)\frac{\left(\int_V \lambda(x,y,z)\,\mathrm{d}v\right)^r}{r!}$$

and (ii) 'none of the r flaws will initiate failure', the probability of which is $[1-F_c]^r$. The event *no failure will be initiated in the volume V*, is the union of disjoint events characterised by probabilities $p^0_{(r)}$ and its probability p^0, according to the total probability theorem, is

$$p^0 = \sum_{r=0}^{\infty} p^0_{(r)} = \exp\left(-\int_V \lambda(x,y,z)\,\mathrm{d}v\right)\sum_{r=0}^{\infty}\frac{\left([1-F_c]\int_V \lambda(x,y,z)\,\mathrm{d}v\right)^r}{r!} \tag{9}$$

Since

$$\sum_{r=0}^{\infty}\frac{\left([1-F_c]\int_V \lambda(x,y,z)\,\mathrm{d}v\right)^r}{r!} = \exp\left([1-F_c]\int_V \lambda(x,y,z)\,\mathrm{d}v\right),$$

Eq. (9) can be simplified to

$$p^0 = \exp\left(-F_c\int_V \lambda(x,y,z)\,\mathrm{d}v\right)$$

and the probability p_f of failure for the component with volume V becomes

$$p_f = 1-\exp\left(-F_c\int_V \lambda(x,y,z)\,\mathrm{d}v\right) \tag{10}$$

Eq. (10) also holds for the two- and one-dimensional case if the volume V is replaced by the area S or the length L of the component. Correspondingly, the flaw number density will be a number of flaws per unit area or unit length.

Since $\overline{\lambda} = \frac{1}{V}\int_V \lambda(x,y,z)\,\mathrm{d}v$ is the expected (average) number density of flaws in the volume V, Eq. (10) can also be presented as

$$p_f = 1-\exp\left(-\overline{\lambda}VF_c\right) \tag{11}$$

A very important special case for the practical applications is obtained when the flaws follow a homogeneous Poisson process in the volume V of the specimen. In this case, the flaws locations are uniformly distributed in the bulk of the component. The defect number density is constant $\lambda(x,y,z) = \lambda = \text{const.}$ and the probability of failure in Eq. (10) becomes

$$p_f = 1-\exp(-\lambda VF_c) \tag{12}$$

Unlike Eqs. (4) and (5), λ in Eq. (12) is the number density of *all* flaws in the stressed volume V and is a measurable quantity.

An upper bound of the probability of failure p_f can be produced if *weak flaws* ($F_c \approx 1$) are assumed. This is a very conservative assumption, suitable in cases where the upper bound of the probability of failure is required.

Eq. (11) can be generalised for multiple type of flaws. Thus, if M different types of flaws are present, the probability that no failure will be initiated is

$$p^0 = \exp(-\overline{\lambda}_1 VF_{1c}) \times \cdots \times \exp(-\overline{\lambda}_M VF_{Mc}) = \exp\left(-V\sum_{i=1}^{M}\overline{\lambda}_i F_{ic}\right)$$

where $\overline{\lambda}_i$ and F_{ic} are the average flaw number density and the conditional individual probability of initiating failure characterising the ith type of flaws. This equation expresses the probability that no failure will be initiated by the first, the second, ..., the Mth type of flaws. The probability of failure then becomes

$$p_f = 1 - \exp\left(-V\sum_{i=1}^{M}\overline{\lambda}_i F_{ic}\right) \tag{13}$$

In order to distinguish between a complex stress state and a uniaxial stress state, for a volume V subjected to a uniaxial stress σ, the probability F_c in Eq. (11) will be denoted by $F(\sigma)$.

3. Determining the conditional individual probability of initiating failure, characterising a single flaw

The conditional individual probability F_c of initiating failure characterising a single flaw can be estimated using a Monte Carlo simulation. Random locations and orientations for the flaw are generated in the volume V of the component, according to the flaw number density $\lambda(x, y, z)$. For the important special case where the flaw number density is constant throughout the volume $\lambda = \text{const.}$, the generated random locations should be uniformly distributed in the volume V. For each random location and orientation, a random flaw size is generated by sampling the size distribution of the flaws. Given the specified location, orientation and size of the flaw, a failure criterion is applied to check whether the flaw will be unstable (will initiate failure).

Eq. (11) is very flexible and general because it permits the conditional individual probability F_c of initiating failure to be estimated using different methods. Indeed, the failure criterion is not restricted to fracture mechanics criteria only. It can also be based on other models related to the micromechanics of initiating failure. For the special case of brittle fracture and flaws whose shape can be approximated well by penny-shaped cracks for example, a mixed-mode coplanar strain-energy release rate criterion (Paris and Sih, 1965):

$$G = \frac{(1-v^2)K_I^2}{E} + \frac{(1-v^2)K_{II}^2}{E} + \frac{(1+v)K_{III}^2}{E} \tag{14}$$

can be used (Evans, 1978).

In Eq. (14), G is the strain energy release rate; K_I, K_{II} and K_{III} are the three stress-intensity factors corresponding to the three basic loading modes which are functions of the stress magnitude and crack geometry; E is the elastic modulus and v is the Poisson ratio. Fracture, according to this criterion occurs if the value of the strain energy release rate G exceeds the critical strain energy release rate G_c for the material. This criterion is based on the assumption that planar penny-shaped cracks propagate along their initial planes if $G > G_c$ is fulfilled.

The conditional individual probability F_c of initiating failure characterising a single flaw is estimated by dividing the number of simulations in which failure has been initiated to the total number of Monte Carlo trials. Finally, substituting the estimate F_c in Eq. (11) yields the probability of failure of the stressed component, *irrespective of its geometry*, *type of loading and flaw number density*! The algorithm in pseudocode is given in Appendix A.

Using this algorithm, for different loading levels, the lower tail of the strength distribution for any loaded component with internal flaws can be constructed. For a specified time interval, plugging the strength distribution into the overstress reliability integral (Todinov, 2004) yields the reliability of the component associated with an overstress failure mode.

The efficiency of the algorithm can be increased significantly if the loaded component is divided into N sub-volumes. If a finite element solution is used, the sub-volumes are simply the finite elements which partition the volume of the component.

In case of flaws following a homogeneous Poisson process, in order to generate a random flaw location, a sub-volume is randomly selected first, with probability proportional to its volume fraction (Fig. 2).

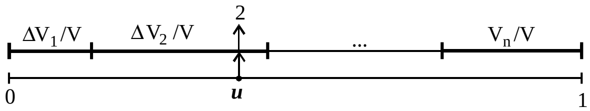

Fig. 2. Random selection of a finite element (sub-volume) where the flaw resides.

The discrete distribution specifying the probabilities with which the finite element is selected is

X	1	2	...	N
$P(X=x)$	$\Delta V_1/V$	$\Delta V_2/V$	...	$\Delta V_N/V$

where $X = 1, 2, \ldots, N$ is the index of the sub-volume, ΔV_X is its volume and V is the total volume of the component. The probability with which the ith sub-volume is selected is proportional to its volume fraction $\Delta V_i/V$. The algorithm for selecting a random sub-volume therefore consists of the following steps:

(i) Construct the cumulative distribution

$$P(X \leqslant k) \equiv F(k) = \sum_{i \leqslant k} \Delta V_i/V$$

of the random variable X (the index of the selected sub-volume);

(ii) Generate an uniformly distributed random number u in the interval $(0,1)$; (iii) If $u \leqslant F(1) = \Delta V_1/V$, the first sub-volume is selected, else if $F(k-1) < u \leqslant F(k)$, the kth sub-volume is selected (Fig. 2).

Once a sub-volume has been selected, a defect location is generated inside and the principal stresses at this location are calculated (using interpolation in the case of finite elements). In case of a stress state obtained by using the method of finite elements, the calculation speed can further be increased at the expense of a slight decrease in the calculation precision if another type of approximation is used. Instead of generating a location for the flaw inside the randomly selected finite element and calculating the principal stresses at that location, the principal stresses in the center of the finite element are used. Consequently, no flaw locations inside the randomly selected finite elements are generated and since most finite element solvers provide information regarding the three principal stresses at the center of the finite elements, the speed of computation is increased significantly.

It must be pointed out, that although Eq. (11) gives the probability of failure for the component, it does not reveal the distribution of the locations where failure will be initiated most frequently. Failure will be initiated most frequently in the highest stressed regions where the conditions for a flaw instability will be met first during over-loading.

If in the highest stressed region, no flaw with appropriate type, orientation and size for initiating failure is present, failure will be initiated in a region with lower stress, where an appropriate combination of stress, flaw type, orientation and size exists. The proposed model is precise for loaded components with flaws characterised by a relatively small number density because in this case, the assumption of 'non-interacting' flaws will be closely matched.

Eq. (11) is valid for *an arbitrarily loaded component*, with *complex shape* and non-homogeneous distribution of the flaws. The power of the equation is in relating in a simple fashion the individual probability of failure F_c characterising a single flaw (with locations following the specified non-homogeneous flaw number density $\lambda(x,y,z)$) to the probability of failure p_f characterising the whole population of flaws.

Suppose that a direct Monte Carlo simulation was used to determine the probability of failure of the component. In this case, at each simulation trial, a large number of flaws need to be generated and for each flaw, a check needs to be performed to determine whether there will be at least a single unstable flaw which initiates failure. If Eq. (11) is used to determine the probability of failure of the component, only a single simulation trial involving a single act of generating flaws in the component volume would be necessary. The purpose is to collect statistical information from all parts of the volume stressed in different ways, necessary to estimate the conditional individual probability F_c. Once F_c has been estimated, it is simply plugged into Eq. (11) to determine the probability of failure of the component.

Because only a single simulation trial is involved instead of thousands or millions of trials, the calculation speed of the proposed algorithm is significantly larger than the calculation speed of the direct simulation. Using this procedure, for different loading levels, the lower tail of the strength distribution of any loaded component with internal flaws can be constructed.

It is important to point out that F_c incorporates the influence of the particular loading (stress) state throughout the entire volume of the component. If the stress state in the loaded component is altered, F_c will be altered too despite the fact that all locations, orientations and flaw sizes will remain the same. Another important feature of F_c which distinguishes it from the probability of failure p_f is that while p_f is an absolute probability, F_c is a *conditional probability*. It is the probability that a flaw will cause failure, *given* that it is already inside the volume of the stressed component. By 'moving' the flaw randomly inside the component and by simultaneously changing its shape and orientation, statistical information regarding the conditional probability F_c is gathered.

In effect, Eqs. (10)–(12) constitute the core of a new theory of failure initiated by flaws. It avoids overly conservative estimates for the probability of failure, which result from equating the probability that a flaw will initiate failure in a stressed region with the probability that the flaw will reside in the region. The new concept 'conditional individual probability of initiating failure' characterising a single flaw acknowledges the fact that not all flaws present in the material will initiate failure. In other words, flaws initiate failure with certain probability.

Important application areas of the derived equation are (i) determining the lower tail of the strength distribution for components containing flaws and (ii) assessing the vulnerability of designs to failure initiated by flaws. An application of Eq. (11) and the algorithm in case of fracture caused by sharp penny-shaped cracks will be published elsewhere.

4. Statistics of failure initiated by flaws

The product $\lambda' = \lambda F_c$ in Eq. (12), which we refer to as *detrimental factor*, is an important parameter. Consider for example two components with identical material and geometry. One of the components is characterised by flaws with a high number density λ_1 which initiate failure with small probability F_{c1} and the other component is characterised by flaws with a low number density λ_2 which initiate failure with large probability F_{c2}. If both components are characterised by the same detrimental factors ($\lambda_1 F_{c1} = \lambda_2 F_{c2}$), the probabilities of failure initiated by flaws for both components will be the same.

Eq. (13) shows that the most dangerous type of flaws is the one characterised by the largest *detrimental factor* $\overline{\lambda}_i F_{ic}$. Consequently, the efforts towards eliminating flaws from the material should concentrate on types of flaws with large detrimental factors.

For a uniaxial stress σ and very weak flaws which initiate failure easily, the conditional individual probability of initiating failure can be assumed to be unity $F(\sigma) = 1$. In this case, the probability of failure $p_f = 1 - \exp(-\lambda V)$ of the stressed volume V equals the probability that at least one weak flaw will be present in it. In the general case however, the conditional individual probability F_c of initiating failure characterising a single flaw will be a number between zero and unity. Consequently, Eq. (11) avoids overly conservative predictions regarding the probability of failure of the component.

From Eq. (11), it follows that the smaller the stressed volume V, the smaller the probability of failure.

This is one of the reasons why between two similar components, made of the same material, the larger component is weaker.

The Weibull distribution (3) can be obtained as a special case of Eq. (11). Indeed, if the conditional individual probability of triggering failure at the stress level σ can be approximated by the power law

$$F_c(\sigma) = \frac{1}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^m \tag{15}$$

where V_0 and σ_0 are constants, the substitution in Eq. (11) gives the Weibull distribution (3). In other words, for material with flaws, whose conditional individual probabilities of initiating failure increase with the applied stress according to the power law (15), the probability of failure is described by the Weibull distribution. If the dependence $F_c(\sigma)$ is different from Eq. (15) however, a function different from the Weibull distribution will be obtained after the substitution in Eq. (11). Indeed, suppose that $F(\sigma)$ is described by the monotonically increasing function $F(\sigma) = 1 - \exp\{-k\sigma^r\}$. After the substitution in Eq. (11), the probability of failure becomes

$$p_\sigma = 1 - \exp(-\lambda V) \times \exp[\lambda V \times \exp(-k\sigma^r)]$$

which is not a Weibull distribution.

5. Limiting the probability of failure and decreasing the vulnerability of designs to failure caused by flaws

By solving Eq. (11) numerically with respect to $\bar{\lambda}$ (given a specified maximum acceptable probability of failure $p_{\mathrm{f\,max}}$), an upper bound $\bar{\lambda}_{\mathrm{u}}$ of the average flaw number density upper bound can be determined:

$$\bar{\lambda}_{\mathrm{u}} = -\frac{1}{VF_{\mathrm{c}}} \ln(1 - p_{\mathrm{f\,max}}) \tag{16}$$

This upper bound guarantees that whenever the average flaw number density $\bar{\lambda}$ satisfies $\bar{\lambda} \leqslant \bar{\lambda}_{\mathrm{u}}$, the probability of failure of the component will be smaller than $p_{\mathrm{f\,max}}$.

Fig. 3 gives the dependence between the flaw number density upper bound λ_{u} and $p_{\mathrm{f\,max}}$, for different values of the stressed volume V, in case of very weak flaws ($F_{\mathrm{c}} = 1$).

Consider now a component with volume V, which has been cut from material with flaw number density λ and subjected to a uniaxial stress σ. It is assumed that the flaws, whose locations follow a homogeneous Poisson process, are from a single type. Suppose that failure is controlled solely by the size of the flaws in the material and does not depend on their orientation and shape. The size distribution $G(d)$ of the flaws is the probability $G(d) = P(D \leqslant d)$ that the size D of a flaw will not be greater than a specified value d. Let d_σ denote the critical flaw size for the stress level σ. In other words, a flaw with size greater than the critical size d_σ will initiate failure at a stress level σ.

Given the size distribution of the flaws, we can determine the maximum acceptable value V of the stressed volume that limits the probability of failure below a maximum acceptable level.

In case of failure controlled solely by the size of the flaws, $F(\sigma)$ in Eq. (6) becomes $1 - G(d_\sigma)$ which is the probability that a flaw will initiate failure at the stress level σ. Substituting $F(\sigma) = 1 - G(d_\sigma)$ in Eq. (6) gives

$$p_\sigma = 1 - \exp\{-\lambda V[1 - G(d_\sigma)]\} \tag{17}$$

for the probability p_σ of initiating failure at a stress level σ.

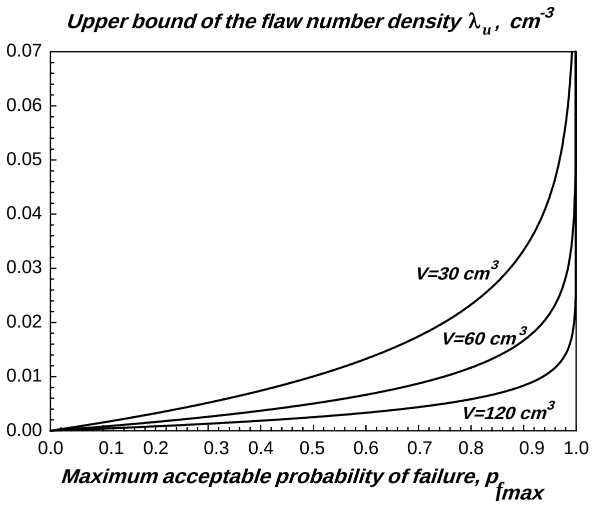

Fig. 3. A flaw number density upper bound, as a function of the maximum acceptable probability of failure, for different values of the stressed volume.

Eq. (17) can be used for calculating the probabilities of failure from the lower tail of the strength distribution in case of failure controlled by the size of the flaws. Limiting the size of the stressed volume limits the probability of failure initiated by flaws, which is of significant importance to the design for reliability. By solving Eq. (17) with respect to V (given a specified maximum acceptable probability of failure $p_{\sigma\,\max}$ at a stress level σ), an upper bound V^* for the stressed volume can be determined:

$$V^* = -\frac{1}{\lambda[1 - G(d_\sigma)]}\ln(1 - p_{\sigma\,\max}) \tag{18}$$

The upper bound V^* guarantees that if for the stressed volume, $V \leqslant V^*$ is satisfied, the probability of failure p_σ will be smaller than the maximum acceptable level $p_{\sigma\,\max}$.

6. Optimising designs by decreasing their vulnerability to failure caused by flaws during overloading

From Eq. (12) it is clear that given the volume of the component, the probability of failure p_f during overloading can be minimised by minimising the detrimental factor λF_c associated with the flaws. In case of a large flaw number density λ, the probability of failure p_f is very sensitive to the conditional individual probability of failure F_c and relatively insensitive to the number density of the flaws λ. Consequently, a significant reduction of the probability of failure can be achieved by a slight reduction of the conditional individual probability of failure F_c. Conversely, in case of a large conditional individual probability of failure, the probability of failure becomes sensitive to the flaw number density and relatively insensitive to the conditional individual probability of failure. Consequently, an efficient reduction of the probability of failure can be achieved by reducing the flaw number density. The decision about which method of reduction for the probability of failure should be preferred depends also on the balance between the cost of investment and the actual risk reduction associated with it. If the cost of investment towards the risk reduction outweighs the benefit from the risk reduction, no action is taken. If the benefit from the risk reduction however outweighs the cost of investment towards it, measures are implemented to reduce the risk.

As can also be verified from Eq. (12) given the volume of the component, the size distribution of the flaws and their number density, minimising the probability of failure requires minimising the conditional individual probability of failure F_c. The advantage of the new equation for decreasing the vulnerability of designs to failure caused by flaws can be illustrated by the following simple example.

A solid bar with length L and constant cross-section S (Fig. 4a) contains flaws whose locations in the volume of the bar follow a homogeneous Poisson process, with a constant flaw number density λ and size distribution according to Fig. 4b. The bar is firmly supported (at a point A in Fig. 4a) at a distance x from its left end. There exists also a chance of an excessive overload in axial direction. Given that overloading of the bar is

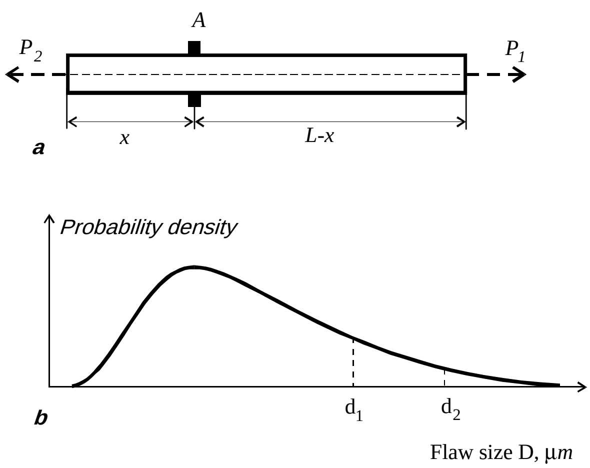

Fig. 4. (a) A solid bar loaded in tension by dynamic forces; (b) Size distribution of the flaws in the bar.

present, there exists a probability q that a dynamic force of magnitude P_1 will overload the bar in tension right from the support and a probability $1 - q$ that a dynamic force of magnitude $P_2 < P_1$ will overload the bar in tension left from the support. Suppose that if the bar is overloaded in tension by a dynamic force of magnitude P_1, any flaw with size greater than the critical value d_1 (Fig. 4b) will cause failure. The probability $P(D > d_1) = \alpha_1$ of having a flaw with size D greater than d_1 is equal to the area α_1 beneath the upper tail of the probability density distribution of the flaw size in Fig. 4b, located to the right of d_1. Accordingly, if the bar is overloaded in tension by the dynamic force P_2, any flaw with size D greater than the critical value d_2 ($d_2 > d_1$) will cause failure. The probability $P(D > d_2) = \alpha_2$ that a randomly selected flaw will have a size greater than d_2, is equal to the area α_2 beneath the upper tail of the probability density distribution located to the right from point d_2 in Fig. 4b.

Given that overloading is present, according to the total probability theorem, the conditional individual probability of failure associated with a single flaw is

$$F_c = (x/L)(1 - q)\alpha_2 + (1 - x/L)q\alpha_1 \tag{19}$$

In Eq. (19), x/L is the probability that a single flaw with a random location existing with certainty in the volume of the bar, will be on the left side of the support; $(1 - x/L)$ is the probability that the flaw will be on the right side of the support. Given that an overloading is present, the term $(x/L)(1 - q)\alpha_2$ in Eq. (19) is the probability that failure will be initiated left from the support and the term $(1 - x/L)q\alpha_1$ is the probability that failure will be initiated right from the support. Substituting the values in Eq. (12), the probability of failure of the bar given that overloading is present becomes:

$$p_f = 1 - \exp(-\lambda LS[(x/L)(1 - q)\alpha_2 + (1 - x/L)q\alpha_1]) \tag{20}$$

An important consideration during selecting the location of the support is selecting its distance x in such a way that the probability of failure triggered by flaws in case of overloading is minimised. Clearly, this is achieved when the conditional individual probability of failure F_c in Eq. (19) is minimised. Since F_c in Eq. (19) is a linear function of x, the minimum is attained when either $x = 0$ or $x = L$. Since $F_{c|x=0} = q\alpha_1$ and $F_{c|x=L} = (1 - q)\alpha_2$, if $q\alpha_1 < (1 - q)\alpha_2$ the support location minimising the probability of failure is at the left end of the bar. If $q\alpha_1 > (1 - q)\alpha_2$, the support location minimising the probability of failure is at the right end of the bar. Finally, if $q\alpha_1 = (1 - q)\alpha_2$, the support could be anywhere along the bar because, in this case, the conditional individual probability of failure is the same. Interestingly, if $q\alpha_1 \neq (1 - q)\alpha_2$ the bar is least vulnerable to failure caused by flaws when the support is located at one of the ends, irrespective of the numerical values of the controlling parameters L, S, λ, α_1, α_2 and q.

The parameters λ and the size distribution in Fig. 4 can be determined using X-ray or ultrasonic methods which allows us to calculate the probability of failure of the bar. It is not clear however how can the probability of failure of the bar be calculated by using Eqs. (4) or (5). The expected number of critical defects in Eq. (5) is not a measurable quantity. What is being measured using methods from the quantitative metallography is the actual number of flaws and the actual flaw size distribution.

7. A stochastic model related to the fatigue life distribution of a component containing defects

An equation similar to Eq. (11) can be developed for determining the fatigue life distribution for a loaded component whose surface contains manufacturing defects or defects caused by a mechanical damage, with a specified number density, geometry and size distribution. The model is based on (i) the concept conditional individual probability that the fatigue life associated with a single defect will be smaller than a specified value *given* that the defect is on the stressed surface, (ii) a model relating this conditional probability to the unconditional probability that the fatigue life of a component containing a population of defects will be smaller than a specified value and (iii) the stress field of the loaded surface, determined by an analytical or numerical method.

Suppose that a component with complex geometry is fatigue loaded in an arbitrary fashion, and contains non-interacting surface flaws. It is assumed that the flaws locations on the surface of the component with total area S follow a non-homogeneous Poisson process. The variation of the defect number density on the surface of the component is described by the function $\lambda(x,y)$ which gives the defect number density in the infinitesimal surface element ds at a location with coordinates x, y.

Let $Q_c(n)$ denote the conditional individual probability (the index 'c' stands for 'conditional') that the fatigue life characterising a single defect with location following the flaw number density $\lambda(x,y)$ on the component's surface will be smaller than n cycles, *given* that the defect resides on the surface. This probability is different from the probability $F(n)$ that the fatigue life of the component (whose failure is caused by a surface defect from a population of surface defects) will be smaller than n cycles. The probability $F(n)$ is related to the whole population of defects and is not conditioned on the existence of defects on the surface of the component. In other words, $F(n)$ is still a valid concept even if defects are not present at all on the surface.

The probability $p^0_{(r)}$ of the compound event: *exactly r defects reside on the surface of the component and none of their fatigue lives will be smaller than n cycles* can be presented as a product

$$p^0_{(r)} = \exp\left(-\int_S \lambda(x,y)\,\mathrm{d}s\right)\frac{\left(\int_S \lambda(x,y)\,\mathrm{d}s\right)^r}{r!}[1-Q_c(n)]^r \tag{21}$$

of the probabilities of two statistically independent events: (i) *exactly r defects reside on the surface S*, the probability of which is

$$P(X=r) = \exp\left(-\int_S \lambda(x,y)\,\mathrm{d}s\right)\frac{\left(\int_S \lambda(x,y)\,\mathrm{d}s\right)^r}{r!}$$

and (ii) *none of the fatigue lives associated with the r defects will be smaller than n cycles*, the probability of which is $[1-Q_c(n)]^r$. The event *component's fatigue life will be greater than n cycles* is the union of the disjoint events characterised by probabilities $p^0_{(r)}$ and its probability p^0, according to the total probability theorem, is

$$p^0 = \sum_{r=0}^{\infty} p^0_{(r)} = \exp\left(-\int_S \lambda(x,y)\,\mathrm{d}s\right)\sum_{r=0}^{\infty}\frac{\left([1-Q_c(n)]\int_S \lambda(x,y)\,\mathrm{d}s\right)^r}{r!} \tag{22}$$

Since

$$\sum_{r=0}^{\infty}\frac{\left([1-Q_c(n)]\int_S \lambda(x,y)\,\mathrm{d}s\right)^r}{r!} = \exp\left([1-Q_c(n)]\int_S \lambda(x,y)\,\mathrm{d}s\right),$$

Eq. (22) can be simplified to

$$p^0 = \exp\left(-Q_c(n)\int_S \lambda(x,y)\,\mathrm{d}s\right)$$

The probability $F(n)$ that the fatigue life of the component will be smaller than n cycles is equal to the probability that on the component's surface there will be at least one defect with fatigue life smaller than n cycles. Accordingly,

$$F(n) = 1 - \exp\left[-Q_c(n)\int_S \lambda(x,y)\,\mathrm{d}s\right] \tag{23}$$

Since $\overline{\lambda} = \frac{1}{S}\int_S \lambda(x,y)\,\mathrm{d}s$ is the expected (average) number density of the defects on the surface S, Eq. (23) can also be presented as

$$F(n) = 1 - \exp\left(-\overline{\lambda}SQ_c(n)\right) \tag{24}$$

An important special case of Eq. (23) can be derived for defects following a homogeneous Poisson process on the surface S. In this case, the defect number density is constant $\lambda(x,y) = \lambda =$ const. and the probability that the fatigue life will be smaller than a specified number n of cycles becomes

$$F(n) = 1 - \exp\left(-\lambda SQ_c(n)\right) \tag{25}$$

The conditional probability $Q_c(n)$ related to a single defect can be estimated using a Monte Carlo simulation, similar to the way the conditional probability F_c in Eq. (11) was estimated.

A single defect with a random orientation, shape, size and location following the specified number density $\lambda(x,y)$ is generated on the surface S of the loaded component. Next, for each generated location, orientation and size of the defect, the fatigue life is estimated. $Q_c(n)$ is obtained as a ratio of the number of defect locations for which the predicted fatigue life was smaller than or equal to n cycles and the total number of simulation trials. In this way, statistical information related to a single defect is collected first from different parts of the stressed surface. If the stress state is altered, the conditional probability $Q_c(n)$ is also altered.

Substituting the estimated conditional probability $Q_c(n)$ in Eq. (25) yields the probability of fatigue failure $F(n)$ before n cycles.

The stress tensor, stress range and the mean stress characterising different locations of the flaw on the stressed surface can be obtained from a finite element analysis. In case of flaws following a homogeneous Poisson process, the stressed surface can be partitioned into finite elements and instead of generating random locations for the defects, the finite elements can be randomly selected with probability proportional to their areal fraction on the surface. After the selection of a finite element, a random location of the flaw can be selected uniformly distributed inside the element.

Similar to the overstress failure model, in case of a stress distribution on the surface obtained by using the method of finite elements, the calculation speed can further be increased at the expense of a slight decrease in the calculation precision if an approximation is used. Instead of generating a location for the defect in the randomly selected finite element and calculating the principal stresses at that location, the principal stresses in the center of the finite element are used which are readily available from the file produced by the finite elements solver. As a result, no defect locations inside the randomly selected finite elements are generated and the speed of computation is increased significantly.

Parametric studies based on this stochastic model can be conducted to explore the influence of the uncertainty associated with factors such as shape, size, number density of defects and associated residual stress fields, on the confidence levels of the fatigue life predictions. The stochastic model will be an excellent basis for specifying the maximum acceptable level of the defects number density which guarantees that the risk of fatigue failure remains below a maximum acceptable level.

Another important application of the model is in optimising designs and loading in order to minimise the probability of fatigue failure initiated by defects. In effect, this is a way to decrease the vulnerability of designs to fatigue failure, initiated by surface flaws.

Similar to Eq. (11) proposed for the case of an overstress failure of a loaded component, Eqs. (23)–(25) avoid overly conservative predictions related to the length of fatigue life. The reason is that the equations are based on recognising the fact that not all defects in the stressed region will evolve into propagating fatigue cracks. In other words, defects initiate propagating fatigue cracks with certain probability.

Calculating the probability of fatigue crack initiation for a particular combination of random defect size, orientation, and location characterised by a particular stress tensor, incorporates models and experimental data related to the micromechanics of initiating fatigue cracks (Jiang and Sehitoglu, 1999; Ringsberg et al., 2000; Wilkinson, 2001).

Eq. (25) can also serve as a basis for specifying the maximum acceptable defect number density which guarantees that the risk of fatigue failure remains below a maximum acceptable level.

8. Conclusions

1. Powerful equations and a fast algorithm have been proposed for determining the probability of failure of components with complex shape containing multiple flaws. The equations are based on the concept 'conditional individual probability of initiating failure' characterising a single flaw *given* that it is in the stressed component.
2. The derived equations constitute the core of a new theory of failure for components with internal flaws. An important application of the equations is in optimising designs by reducing their vulnerability to overstress failure or fatigue failure initiated by flaws.
3. The proposed approach is an alternative to existing overly conservative approaches for predicting the probability of fracture and fatigue failure.

4. Models have been proposed for specifying the maximum acceptable flaw number density and size of the stressed volume which limit the probability of failure.
5. An important parameter referred to as *detrimental factor* has been introduced to characterise components containing flaws.

Appendix A

An algorithm for Monte Carlo evaluation of the probability of failure of a loaded component with flaws following a homogeneous Poisson process

procedure **Calculate_stress_distribution()**
{/* *Calculates the distribution of stresses in the loaded component using analytical solution or a Finite Elements solution. In case of a Finite Element solution, the stress field is determined for a set of finite elements (sub-volumes)* */}.

procedure **Calculate_principal_stresses()**
{/* *Calculates the magnitude and the direction of the principal stresses at the flaw location* */}

procedure **Select_a_random_finite_element()**
{/* *A random sub-volume is selected with probability proportional to its size* */}

procedure **Select_a_random_location_in_the_element()**
{/* *A random, uniformly distributed location is selected in the selected finite element* */.}

procedure **Interpolate_principal_stresses()**
{/* *Interpolates the principal stresses associated with the random locations in the selected finite elements* */}

function **Generate_random_flaw_size()**
{/* *Samples the size distribution of flaws and returns a random flaw size* */}

procedure **Generate_random_flaw_orientation()**
{/* *Generates the cosine directors of a randomly oriented flaw in space, with respect to the directions of the principal normal stresses* */}

procedure **Generate_random_flaw_location()**
{/* *Generates a point with uniformly distributed coordinates* (x, y, z) *in the volume of the component* */}

function **Check_for_failure_initiation()**
{/* *Uses a failure criterion to check whether the flaw is unstable and returns TRUE if the flaw with the selected location, size and orientation initiates failure* */}

Failure_counter = 0;
Calculate_stress_distribution();
For $i = 1$ **to** Number_of_trials **do**
{
Generate_random_flaw_size ();
Generate_random_flaw_orientation();
In case of analytical solution for the distribution stresses in the component:
{
Generate_random_flaw_location();
Calculate_principal_stresses ();
}
In case of a finite element solution:
{

```
    Select_a_random_finite_element();
    Select_a_random_location_in_the_element();
    Interpolate_principal_stresses();
   }
    Unstable = Check_for_failure_initiation();
    If (Unstable) then Failure_Counter = Failure_Counter + 1;
  }
 F_c = Failure_counter/Number_of_trials;
 Probability_of_component_failure = 1 − exp(−λVF_c).
```

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